



ARTS1030

: cfa g c Z K f]h]b[. @]hYfUhi fYž; YbfYž7i `hi fY

HYfa 'C bY''##'&\$%

7ci fgY 8YW] g

7 fYX]hDc]bhg * *

Gi a a UfmcZh\Y 7ci fgY

Gi VYWh5fYU. 7 Y feZ

 $H\ [g'Wc'\ fgY'WUb'U'gc'VY'gh'\ X]YX']b'h\ Y'Zc``ck\]b[\ 'gdYW]U`]gUh]cb.\ '5\ d\ MSf[hWI\ \ d[f[\ Y]\ MSf[hWI\ \]b']b'h\ MSf[hWI\ \ d[\ Y]\ MSf[hWI\ \]b'h\ MSf[hWI\ \ \]b'h\$

H\]g`Wci fgY`]bHfcXi WYg`mci 'hc`9b[`]g\ 'h\fci [\'h\Y`ghi XmcZ`]hYfUfmiZcfa ": cW g]b[`cb`h\Y`a U'cf``]hYfUfm [YbfYg`cZdcYhfmždfcgY`UbX`XfUa Už'h\Y`Wci fgY`k]``Yl Ua]bY`k Umg`cZk f]h]b[`]b`9b[`]g\ 'Zfca 'U'fUb[Y`cZ dYf]cXg`UbX`W` 'hi fU``HfUX]h]cbg`]bWi X]b[`bcb!6f]h]g\ `UbX`dfY!Hk Ybh]Yh\ 'WYbhi fm'' \dik]```U`gc`]bHfcXi WY mci 'hc`h\Y`gdYW]u`]gYX`j cWJVi `UfmcZ`]hYfUfmUbX`hYl hi U``UbU`mg]g"

7 ci fgY @YUfb]b[Ci hWca Yg

&" 7 f]h]WU``mUbU`mgY ``]hYfUfmihYI hg`]b hYfa g`cZei Ygh]cbg`cZ`]hYfUfmZcfa

' " 7 cbXi WiVUg]WfYgYUfW. cb``]MfUfmiYI hgži g]b['Uddfcdf]UHY gW.c`Uf`mfYgci fWg

("5fh]W``UhY`UbX`WcbhYI hi U`]gY`mci f`ck b`Wf]h]WU``dcg]h]cb

) " 7 cbg/fi WhubX:Uff]W `UhY:U:Wf]h]WU:"Uf[i a Ybhcfu``mubX:]b:k f]h]b[

*" I gY 'Uddfcdf]UhY 'Zcfa Uhg ']bWi X]b['WcffYWhfYZYfYbW]b['hc 'dfYgYbh'mci f'k cf_

HYUW]b[GhfUhY[]Yg

H\Y'Wci fgY']g'HJi [\hih\fci [\'``YWfi fYg'UbX'hi hcf]U`g'UbX'cbY'\ci f'cZglfi Wfi fYX'cb`]bY'`YUfb]b[''UWfj]]mf

@YWhifYq.

H\Y "YWi fYg" UfY "XYg][bYX 'rc ']bffcXi W 'gh XYbhg 'rc 'h\Y ']ggi Yg 'UbX 'hY l hg 'cZh\Y 'Wci fgY 'Vmdfcj]X]b[
WcbhYl hi U "]bZcfa Uh]cbz'ci h]bYg 'cZWf]h]WU "fYgdcbgYg 'rc 'h\Y 'hY l hg 'UbX 'gca Y 'WcgY 'X]gW gg]cb" "@YWi fYg
U'gc 'a cXY "Zcf'gh XYbhg 'h\Y 'k Umg']bZcfa Uh]cb ']g 'cf[Ub]nYX 'UbX 'dfYgYbhYX']b 'h\Y 'X]gW]d `]bY 'cZ9b[`]g\"
H\YfY 'k] "'VY 'cddcfhi b]h]Yg ']b 'Yj Yfm YWi fY 'Zcf'ei Ygh]cbg 'UbX '[YbYfU "X]gW gg]cb" '7 Yffu]b 'gWXYXi 'YX

'YWi fYg 'UXXfYgg 'h\Y 'gdYWJZ]Wg 'cZUbU mg]gz Uf[i a YbhUh]cb 'UbX 'k f]h]b[']b 'Vch\ 'YggUmg 'UbX 'Yl Ua g" 'H\Y

'YWi fY 'Zcfa Uh]g ']a dcfhUbh]b 'h\Uh]hXYa cbghfUhYg 'Zcfa U 'a cXYg 'cZdfYgYbh]b['U'ghfi Wi fYX 'Uf[i a Ybhz UbX

Zcf'h\]g 'fYUgcb "YWi fY 'UhhYbXUbW"]g 'Wca di 'gcfm' '9UW\ "YWi fY 'k]" "]bWi XY 'U'g\cfhcb`]bY 'ei]n 'k \]W\ 'a i gh

VY 'Wca d'YhYX 'Xi f]b['U'gdYWJZ]YX 'h]a Y 'Xi f]b['h\Y "YWi fY" 'H\YgY'ei]n 'fYgdcbgYg'k]\" 'dfcj]XY 'U'glUfh]b[
dc]bhZcf'X]gW gg]cb ']b 'h 'rcf]U'g" 'H]a YhUV Y 'WUg\Yg'a i ghVY ']bX]j]Xi U "mUddfcj YX 'Vmh\Y 'Wci fgY

Wcbj Ybcfz'UbX 'UfY 'XYdYbXYbhcb' WYfh]b _Ymi bXYfhU_]b[g'Vmghi XYbhg" '9Wc'' *\$ 'gi ddcfh'UbX

Dck YfDc]bhbchYg' UfY 'dfcj]XYX'j]U'@A G'hc'gi ddcfhih\Y "YWi fYg'UbX 'Zcf'fYj]g]cbz'bchUg'U'gi Vgh]h 'hY 'Zcf
'YWi fY'UhYbXUbW""

Hi hcf]U`g.

 $H\Y'h hcf]U'g'fYei]fY'gh XYbhg'hc'Yb[U[Y'UWh]j Y'mk]h h Y'YWh fY'UbX'fYUX]b['a UhYf]U'`UbX'XYj Y`cd'h Y]f ck b'Wf]h]WU'`fYgdcbgYg''$

H\]g'k]``'c\W\f'h\fci[\.

- $\bullet \text{````k YY_`mdfYdUfUh]cbž']bWi X]b[\text{`XYj }]g]b[\text{`X]gW} gg]cb\text{`ei Ygh]cbg'}Zcf'h\Y'[\text{fci d'UbX}\#f'fYgdcbgYg'hc``YWi fYei]nnYg}$
- " "glfi Wi fYX"UbX"]bZcfa U`"WUgg"X]gW gg]cb
- ""Wc"UVcfUhjj Y'k cf_']b ga U" [fci dg"

 $5\,\text{thY}\,b\,X\,U\,b\,W\,'\,U\,h\,U\,b\,X\,'\,d\,U\,f\,f\,J\,W\,d\,'\,f\,h\,f\,g\,'\,J\,g\,'\,W\,f\,h\,f\,U\,'\,f\,h\,'\,f\,h\,Y\,'\,X\,Y\,j\,Y\,'\,c\,d\,a\,Y\,b\,h\,c\,Z\,W\,f\,f\,J\,'\,h\,J\,b\,_\,J\,b\,[\,\,\,'\,U\,b\,X\,'\,U\,b\,U\,'\,m\,f\,J\,W\,g\,_\,J\,'\,h\,h\,g\,f\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,f\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,g\,'\,g\,'\,H\,h\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g\,g\,'\,H\,h\,g$

Cb`]bY'@YUfb]b['5Wh]j]hm

5 hth Y 'VY[]bb]b['cZth Y 'Wci fgY 'ghi XYbhg'k]``Wca d'YhY 'U'cbY '\ci f'a UbXUhcfmcb`]bY 'WcgY 'fYUX]b[YI YfW]gY"'H\]g 'YI YfW]gY ']g 'Vch 'X]U[bcgh]WUbX 'Zcfa Uh]j Yž 'UbX 'dfcj]XYg 'h\Y 'cddcfhi b]hm Zcf 'YUf'm ZYYXVUW_Zca 'hi hcfg"'H\Y'UWf]j]hm]g 'bchdUfhcZh\Y'UggYgga Ybh Zcf'h\]g 'Wci fgY"

5ggYgga Ybh

5ggYgga Ybh'HUg_g'

5ggYgga Ybh'hUg_	K Y][\h	8i Y'8UHY	Gh XYbh@YUfb]b[Ci hWca Yg 5ggYggYX
HU_Y! <ca `<="" td="" ua="" y`9i=""><td>) \$1</td><td>\$, #\$) #&\$% `\$*.\$\$`DA</td><td>%ž&ž(ž)</td></ca>) \$1	\$, #\$) #&\$% `\$*.\$\$`DA	%ž&ž(ž)
A U⁄cf'YggUm) \$1	%\$#\$(#&\$%^\$*.\$\$^DA	%ž&Ž ž(ž) ž*

5ggYgga Ybh8YHJ]`g

5ggYgga Ybh'% 'HU_Y!<ca Y'9l Ua

GhUfhiXUhY. '\$*#\$) #&\$%- '\$-.\$\$ '5A

@Yb[h\.'%) \$\$ k cfXg

8YHJ]`g. %) \$\$ k cfXg/bi a Yf]WU` [fUXY dfcj]XYX VmHi fb]h]b"

 $\label{thm:limit} \begin{tabular}{ll} Hi fb]h]b"gYh]b" . "H\]g"Ugg][ba Ybh]g"gi Va]hhYX"h\fci [\ 'Hi fb]h]b"UbX"gh XYbhg"Xc"bchgYY"Hi fb]h]b g]a]"Uf]hmfYdcfhg" \\ \end{tabular}$

5ggYgga Ybh& A U'cf'YggUm

GhUfhXUhY. Bch5dd`]WUV`Y

@Yb[h\.'%) \$\$ k cfXg

8YHJ]`g. %) \$\$ k cfXg/k f]hhYb ZYYXVUW_žUggYgga Ybhfi Vf]WUbX bi a Yf]WU`[fUXY dfcj]XYX VmHi fb]h]b"

 $\label{thm:limit} \begin{tabular}{ll} Hi & fb & fb & gYh &$

5 hyb XUb WY F Yei]fYa Ybhg

5 HHY bXUbVW cZ@YVMi fYg UbX#cf 'Hi hcf]U`g`]g`a UbXUhcfm]b`h\]g`Wci fgY"`I bYI WigYX`UVgYbWY`Zfca 'a cfY`h\Ub &\$I `cZWUggYg`f]Y``YWMi fYg`UbX`hi hcf]U`gŁk]```fYgi `h`]b`h\Y`Uk UfX`U`ZU]``[fUXY"`+b`UXX]h]cb`hc`UhhYbX]b[`Uh `YUgh, \$I `cZ`YWMi fYg`UbX`hi hcf]U`gžmci 'a i ghU`gc`Wca d`YhY`h\Y`Cb`]bY`@YUfb]b[`5WMjj]hmi]b`K YY_g`%`UbX &"

@97 HIF9.: \$\``YV\notation fYg`mcik]\``UV\notation f\color inciUthU]b\'7@C\\%ffYV\color fgY\\color fagY\\color fag

 $\label{eq:hilling_problem} \begin{array}{ll} \text{HI HCF} = & @ \text{`$$} \text{'hi hcf]U'g'mci 'k]} \text{```UWhj Y'mYb[U[Y'k]h\ 'WcfY'Wci fgY'WcbhYbhh\ Uhk]```\Y`d'mci 'hc'UhtU]b'7 @C '(fUfh)W'`UbX'WcbhYl hi U`]gY'mci f'ck b'Wf]h]WU`'dcg]h]cb&UbX'7 @C') 'fWcbghfi WhUbX'Ufh]W'`UhY'U'Wf]h]WU` Uf[i a YbhcfU``mUbX']b'k f]h]b[&" \\ \end{array}$

 $CB@\#9`@95FB\#; ``57H\#\#M``\#``UXX]h]cb`hc``YWh fYg`UbX`hi hcf]U`gžh\]g`Wci fgY`fYei]fYg`mci `hc`Wca d`YhY Ub`cb`]bY`WcgY`fYUX]b[`UWhji]hmXi f]b[`K YY_g`%UbX`&"`H\]g`k]```U``ck `i g`hc`dfcj]XY`YUf`mZYYXVUW_hc mci "`Mci `a i ghWca d`YhY`h\]g`UWhji]hm]b`cfXYf`hc`Wca d`YhY`h\Y`Wci fgYžYj Yb`h\ci [\`]h]g`bchdUfhcZ h\Y`Zcfa U``UggYgga YbhZcf`h\Y`Wci fgY"$

7ci fqY'GWXYXi `Y

J]Yk 'WUgg'ha YHUV'Y

H]a YHJV`Y

8 UhY	HmdY	7 cbh/bh
K YY_'%'%': YVfi Ufm!'&(: YVfi Ufm	@YWn fY	±blfcXi Whjcb'hc``jhYfUfmighi XjYg'UbX'h\Y'dfUWhjWYg'cZ WcgY'fYUXjb["
		@YWnfYf."DfcZYggcf"6f][]HtU"C"iVUg
	Hi hcf]U`	Hcd]W=blfcXi Wcfm
	C b`]bY '5 Wf]j]hm	C b`]bY'7`cgY'F YUX]b['9 YfW]gY
K YY_^&.^&) ': YVfi Ufm!'' A UfVX	@YWn fY	"K \Uh]g'DcYlfm[ccX'Zcf3"
		@YWn fYf. '8 f'GYUb'Dfmcf
	Hi hcf]U`	Hcd]W'DcYlfm
	C b`]bY '5 Wf]j]hm	C b`]bY'7`cgY'F YUX]b['9 YfW]gY
K YY_" ."("A UfVX"! "%\$ A UfVX	@YWn fY	"5Vcf][]bU`DcYlfm"
		; iYgh@YWhifYf.`Agʻ9jY`mbʻ5fU`iYbʻ7cff
	Hi hcf]U`	Hcd]W'5Vcf][]bU`DcYlfm
K YY_^(.^%%A UfWX^!^%+ A UfWX	@YWn fY	±blfcXi Whob ho ho h Y G\cfhGhcfm
		BUa '@Yfg'FZW4aSf
		@YWn fYf. DfcZYggcf'6f][]htU'C`i VUg

1	

FYgci fW/g

DfYgWf]VYX F Ygci fWYg

G9H"H9LHG"!"Mci "g\ci "X"Ybgi fY"mci "\Uj Y"Wcd]Yg"cZh\Y"Zc""ck]b[.

DcYlfm'dcYa g'gYhZcf'gh Xmk]``VY'Uj U]`UV'Y'j]U'h\Y'Wci fgY'A ccX`Y'g]hY

G\cfhGhcfmBUa @YžFZW4aSf

Bcj Y`. 'BY``U'@UfgYbž'BSee[`Y

8 fUa U. K]`]Ua $G U_YgdYUfYž = [Y > WSd$

FYWca a YbXYX FYgci fW/g

A '< '5VfUa qž'3 9 'aeeSck aX>[fWcSck FWd_e

7ci fgY 9j U'i Uh]cb UbX 8Yj Y'cda Ybh

±0.8\$% žighi XYbhgʻhcʻXʻi gʻ]bʻh\Y]fʻWci fgYʻYj Uʻi Uh]cbgʻh\Uhih\YmZci bXʻh\YʻfYei]fYa Ybhhcʻgi Va]hk f]hhYb bchYgʻYUW\ k YY_ʻZcfih hcf]UʻgʻhcʻVYʻei]hYʻcbYfci gʻUbXʻgU]Xʻh\Uhih\Yʻk cf_ʻk Ugʻbch\YʻdZ ``]bʻdfYdUf]b[ʻZcf hi hcf]Uʻgʻʻ±bʻfYgdcbgYžk Yʻ\Uj YʻfYa cj YXʻh\]gʻfYei]fYa YbhʻZca 'h\YʻWci fgYʻUggYgga YbhgžʻUbX]bhfcXi WYXʻcb`]bYʻei]nnYgʻhcʻVYʻWca dʻYhYXʻXi f]b[ʻ`YWhi fYgʻYUW\k YY_''

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Hi fb]h]b Gi Va]gg]cb

_Zmci YbWci bhYf'U'dfcV`Ya 'k \Yb'UhYa dhJb['hc'gi Va]hmci f'Ugg][ba Ybhh\fci [\ 'Hi fb]hJbž'd`YUgY
hY`Yd\cbY'9| hYfbU``Gi ddcfhcb'-',) '' ' ' %cf'Ya U]`'h\Ya 'cb'Y! hYfbU`hY`hgi ddcfh4 i bgk "YXi "Ui '"'Gi ddcfh
\ci fg'UfY', .\$\$Ua '-'%\$.\$\$da 'cb'k YY_XUmg'UbX'-.\$\$Ua '-') .\$\$da 'cb'k YY_YbXg'fl *) 'XUmg'U'mYUft'' =Zmci
UfY'i bUV`Y'hc'gi Va]hmci f'Ugg][ba YbhXi Y'hc'U'ZUi 'hk]h\'Hi fb]hJb'mci 'a Um'Udd'mZcf'Ub'Y! hYbg]cbžVi hmci
a i ghfYhU]b'mci f'hJW_Yhbi a VYf'Zfca '9! hYfbU``Gi ddcfhfU'cb['k]h\'Ubmch\Yf'fY`Yj UbhXcW a Ybhgt'hc
]bWi XY'Ug'Yj]XYbW'hc'gi ddcfhmci f'Y! hYbg]cb'Udd`]WUhJcb'' =Zmci 'Ya U]`'9! hYfbU``Gi ddcfhmci 'k]``

5 WUXYa | W < cbYghmUbX D`U[| Uf]ga

D`U[]Uf]ga i]gii g]b[ih\Yik cfXgicfi]XYUgicZch\YfgiUbXidfYgYbh]b[ih\Ya iUgimci fick b"i±iWUbihU_Yia Ubm Zcfa qžZfca iXYi]VYfUhYiW\YUh|b[ihciUVW]XYbhU`mWcdm]b[iZfca iUigci fWYik]h\ci hUW_bck iYX[Ya Ybh'i

IBGK '[fci dg'd'U[]Uf]ga ']bhc'h\Y'Zc``ck]b['WUhY[cf]Yg.

7 cdm]b[.'i g]b['h\Y'gUa Y'cf'j Yfmg]a]`Uf'k cfXg'hc'h\Y'cf][]bU`hYl h'cf']XYU'k]h\ci h'UW_bck `YX[]b['h\Y gci fW'cf'i g]b['ei chUh]cb'a Uf_g"'H\]g'U`gc'Udd`]Yg'hc']a U[Ygž'Ufh'UbX'XYg][b'dfcYWhgž'Ug'k Y``'Ug dfYgYbhUh]cbg'k \YfY'gca YcbY'dfYgYbhg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci h'WfYX]h'

⇒ Uddfcdf]UhY 'dUfUd\fUg]b[...7\Ub[]b['U'ZYk 'k cfXg'UbX'd\fUgYg'k \]'Y 'a cghmfYhJ]b]b['h\Y'cf][]bU` ghfi Wh fY 'UbX#cf'dfc[fYgg]cb'cZ]XYUg'cZh\Y'cf][]bU'žUbX']bZcfa Uh]cb'k]h\ci hUW_bck `YX[Ya Ybh'H\]g'U`gc Udd`]Yg']b'dfYgYbhUh]cbg'k \YfY'gca YcbY'dUfUd\fUgYg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci hWfYX]hUbX'hc d]YW]b['hc[Yh\Yf'ei chYg'UbX'dUfUd\fUgYg']bhc'U'bYk 'k \c`Yžk]h\ci hUddfcdf]UhY'fYZYfYbW]b["

7 c``i g]cb. 'k cf_]b['k]th 'cth Yfg'Vi hidUgg]b['cZZth Y 'k cf_'Ug'U'dYfgcb'g']bX]j]Xi U``k cf_"'7 c``i g]cb'U`gc]bWi XYg'dfcj]X]b['mci f'k cf_'thc'Ubcth Yf'ghi XYbh'VYZcfY'th Y'Xi Y'XUHYZcf'Zcf'th Y'di fdcgY'cZth Ya

- i bXYfglUbX`mci f`f][\hg`UbX`fYgdcbg]V]`]h]Yg`Ug`U`gh XYbh'Uhi BGK
- VY'Uk UfY'cZd`U[]Uf]ga ž'Wcdmf][\hzil BGK 'Gh XYbh7cXY'cZ7cbXi WhUbX'5WWdhUV`Y'I gY'cZ I BGK '= H'F Ygci fWfg'Dc`]Wh
- VY 'Uk UfY 'cZh\Y 'ghUbXUfXg 'cZVY\Uj]ci f'Yl dYWhYX 'cZYj YfmcbY ']b 'h\Y 'l BGK 'Wcaaib]hm
- `cWUHY 'gYfj]WYg'UbX `]bZcfa UH]cb'UVci hI BGK 'UbX'I BGK '@]VfUfm

5 WUXYa]W = 12cfa Uh]cb

 $: \ cf`YggYbh]U``gh`XYbh`]bZcfa\ Uh]cb`fY`Uh]b[\ `hc.$

fYeiYghg Zcf YlhYbg]cb/

cfYa aYbhU`ah h h ha'7cfYLhYbalchyf b7 lcb'