

# **ARTS3091**

5Xj UbWX A YX]U =ggi Yg

HYfa 'C bY''##'&\$%-

## 7ci fgY<sup>\*</sup>Cj Yfj ]Yk

## GHUZZ:7cbHUWh8YHJ]`g`

'7cbj Ybcfg'

B Ua Y	9a U]`	5j U]`UV]`]hm	@cWJhjcb	D\cbY
5bXfYk 'Aifd\]Y	U"a i fd\]Y4 i bgk "YXi "Ui	Ya U]`	FcVYfh KYVghYf 6i]`X]b[ž'' ‱	- ' , ) ) ) ( ,

## GWXcc`7cbHJWhi±bZcfa Uh]cb

Fcca " ‰ž"Yj Y"" FcVYfhiK YVghYf"6i ]`X]b[

D\cbY.'f\$&Ł'-',)'(,)\*

9a U]`. 'gUa 4 i bgk "YXi "Ui

K YVg]hY. <u>\hdg.##gUa "Ufhg"i bgk "YXi "Ui</u>

## 7ci fgY 8YHJ] g

## 7 fYX]hDc]bhg \*

## Gi a a UfmcZh\Y'7ci fgY

Gi V Y Whis f Y U. 'A Y X] Už'7 i 'hi f Y 'Ub X 'H Y W bc'c [ mB Y k 'a Y X] U'h Y W bc'c [] Y g' W U 'Y b [ Y 'a Ubmc Z h Y '[ ] y b g'c Z W 'hi f U' 'Ub X 'g c W] U' 'd f U W J W g" '5 h h Y 'g Ua Y 'hja Y ž' b Y k 'W 'hi f U' `Ub X 'g c W] U' i g Y g 'c Z a Y X] U 'W U' Y b [ Y 'a i W h f U 'Ub X 'g c W] U' 'd c f Y 'h Y 'b U hi f Y 'c Z] b W Y U g] b [ 'm X m b Ua ] W a Y X] U 'h Y W b c 'c [] Y g ž Ub X 'h Y 'b Y k 'W 'hi f U' 'Ub X 'g c W] U' 'd f U W J W g' U' c b [ g] X Y 'k \] W 'a Y X] U 'h Y W b c 'c [] Y g 'Y j c' j Y " M c i 'k ] `` 'W c b g] X Y f \_ Y m W c b h Y a d c f U f m ] X Y U c i ha Y X] U ž' W 'hi f U' 'Ub X 'g c W] U' 'W U b [ Y " G hi X Y b h g 'k ] `` i b X Y f h \_ Y ' [ i ] X Y X 'Ub X g Y `Z X] f Y W Y X 'f Y g Y U f W \ " 'H c d ] W g Y I Ua ] b Y X 'W c i 'X ] b W i X Y . 'X ] [] h U ' Ub X b Y h c f \_ Y X 'a Y X] U c Z U `` \_] b X g / ] a a Y f g ] Y z ] b h Y f U W J y 'C f 'Ui [ a Y b h Y X 'a Y X] U ' " f Y U' ] h Y g " " / X U h U 'g \ Uf ] b [ / h Y '] a d U W c Z a Y X] U 'h Y W b c c [ m h U \_ Y i d '] b \_ Y m W 'hi f U ` Ub X 'g c W] U 'd f U W J W g z Z f Y I U a d 'Y '\ Y U 'h X z Y X W J c f '[ Y b Y h W ] " b z [ c j Y f b a Y b h z b Y k 'Z c f a g c Z W c a a i b ] h z c f 'g W Y b W g 'q i W . 'U g 'W a Uh c c [ m z b Y i f c g W Y b W c f '[ Y b Y h W g '' ]

## 7 ci fgY @YUfb]b[ 'Ci hWca Yg

%" h\]b\_'WYUfmUVci hh\Y'fc`Y'cZa YX]U'hYW\bc`c[]Yg']b'W'hi fU``UbX'gcW]U``W\Ub[Y &" h\]b\_'WYUfmUVci hh\Y']a dUWicZW\`hi fU``dfUWf]WYg'UbX'gcW]U``bYYXg'i dcb'cb[c]b['a YX]U |bj Ybhcb'UbX'Yj c`i hcb

" []i Y WYUf Yl dfYgg]cb hc mci f ck b Wf]h]WU Yb[U[Ya Ybhk]h h YgY W Ub[Yg"

 $("\t Y'Z fhYf'XYj Y`cdYX'hXY'g_]`g'Zfca 'dfYj ]ci g'Wci fgYg']b'gWXc`Uf'm']bei ]fmž'gdYW]ZJWU``m'Wf]hJWU` hX]b_]b[ ž'UbU`mt]WU``fYUX]b[ 'UbX' k f]h]b[ ž']bXYdYbXYbh`YUfb]b[ 'UbX']bhY``YWfi U``Ui hcbca m'$ 

) " i bXYftU\_Y`]bXYdYbXYbh`YUfb]b[ `UbX`gY`ZX]fYWYX`fYgYUfVX`UWIj ]lmUbX`fYdcft|b[ "

- \*"\Uj Y`XYj Y`cdYX`mci f`UV]`]hmhc`Zc``ck `mci f`ck b`]bhYfYghg`h\fci [\`]b`U`f][cfci g`a UbbYfZhc`W\ccgY WUgY`ghi X]Yg`UbX`Yl Ua d`Yg`Zcf`mci fgY`Zh\Uh'UfY`fY`Yj Ubhhc`h\Y`WcbhYl hcZmci f`ck b``YUfb]b[ž Yj YfmXUm'UbX`k cf\_]b[``]ZY
- $+ "\VUj Y`XYj Y`cdYX`mci f`UV]`]h]Yg`hc`Yb[\U[\Y`]b`Wc``UVcfUh]j Y`fYgYUfW\`UbX`X]gW|gg]cb`hc`U`\][\Yf Y']b`Wc``UVcfUh]j Y`fYgYUfW\`UbX`X]gW|gg]cb`hc`U`\][\Yf Y']b`Wc``UVcfUh]j Y`fYgYUfW\`UbX`X]gW|gg]cb`hc`U`\][\Yf Yf Y']$

## HYUWX]b[ GhfUhY[]Yg

## 5ggYgga Ybh

 $\hspace{1cm} \hspace{1cm} \hspace{1cm$ 

## 5ggYgga Ybh'HUg\_g'

5ggYgga Ybh'HJg_	K Y][\h	8i Y'8UHY	Gh XYbh@YUfb]b[ Ci hWca Ygʻ5ggYggYX
DYfgcbU`'7 ci fgY'5fWX]j Y'UbX F YZYW¶cbg'	) \$1	Bch5dd`]WUV`Y	%ž&ž ž(
: ]bU``9ggUm	) \$1	&*#\$(#&\$%-`\$).\$\$`DA	%ž&ž ž( ž) ž* ž+

### 5ggYgga Ybh8YhJ]`g

5ggYgga Ybh'% 'DYfgcbU` '7ci fgY '5fWX]j Y 'UbX F YZYWh]cbg

GhUfhXUhY. Bch5dd`]WUV`Y

@Yb[h\.`\*'l'(\$\$'kcfX'V'c['dcghg'cf'Yei]jUYbh'fbyYY'AccX'Y'WcifgY'g]hY'Zcf'acfY'XYhU]`gŁ

8YHJ]`g. `\*`I`(\$\$`k cfX`V`c[`dcghg`cb`A ccX`Yž`UggYggYX`hk`]WYž'cbWY`YUf`mik`]h\`Zcfa Uh]j Y`ZYYXVUW<u>"</u>"

5XX]h]cbU`XYhJ]`g.

GYY'h\Y'A ccX`Y'Wci fgY'g]hY'Zcf'a cfY'XYhU]`g"

Gi Va ]gg]cb bchYg. Mci 'bYYX'hc'gi Va ]hV`c[ 'dcghg'Zcf'k YY\_g'&'UbX'' 'VYZcfY'h\Y'h hcf]U`g'Zcf'h\cgY k YY\_g''Mci 'h\Yb'bYYX'hc'W\ccgY'Zci f'k YY\_g'Zfca 'k YY\_g'( 'hc'- 'hc'gi Va ]hV`c[ 'dcghgž'U`k Umg'VYZcfY'h\Y h hcf]U`'Zcf'h\Y'k YY\_']bj c'j YX'''Mci 'a i ghi gY'h\Y'V`c[ 'gcZk UfY'dfcj ]XYX'cb'A ccX'Y'hc'dcgh'

Hi fb]hjb gYhjb[. H\]g ]g bchU Hi fb]hjb Ugg][ ba Ybh

5ggYgga Ybh'& : ]bU` 9ggUm

GhUfh XUhY.

@Yb[ h\.'' \$\$\$'k cfXg'cf'Yei ]j U`Ybh

8YHJ]`g. "\$\$\$`k cfXgž'fi Vf]Wcb`m'ZYYXVUW\_"

5XX]h]cbU`XYhU]`g.

9ggUmhcd]WgrUbXrZ fh\Yfr]bghfi Whjcbgrk]``VYrdcghYXrcbrh\YrWci fgYrA ccX`Yrg]hY"

Gi Va ]gg]cb bchYg.Gi Va ]hj ]U'Hi fb]h]b cb h\Y'Wti fgY'A ccX`Y'g]hY"

 $\label{thm:limit} \begin{tabular}{ll} Hi & fb & fb & gYh &$ 

### 5 hhyb XUb WY F Yei ]fYa Ybhg

#### @YWhi fYg UbX Ch\YfA UhYf]U`g ZcfHi hcf]U`DfYdUfUh]cb

Mci 'a i ghYb[ U[ Y k ]h\ 'U``'h\ Y ``YWi fY `UbX'ch\Yf'a UhYf]U`g'Zcf'h hcf]U``dfYdUfUh]cb`cb`]bY"'Mci 'a i ghXc'h\]g VYZcfY'h\Y'h hcf]U`g'Zcf'h\Y'k YY\_\_']bj c`j YX"'I bYI W gYX``UW\_'cZYb[ U[ Ya Ybhk ]h\ 'a cfY'h\Ub'k c'k YY\_gfi `YWi fYg'UbX'ch\Yf'a UhYf]U`g'Zcf'h hcf]U``dfYdUfUh]cb`k ]``fYgi 'h']b'h\Y'Uk UfX`cZU'ZU]`'[ fUXY"

BchY'h\Uhih\YfY']g'U''YWh fY']b'h\Y''YWh fY'h\YUhfY']b'k YY\_'cbYž'UbX'cb'm]b'k YY\_'cbY"'H\]g']g'gc'h\Uhik Y WUb'a YYhi d'hc[Yh\Yf'cb'U'k\c'Y'Wci fgY'VUg]g'Uhi'YUghcbWY"'5Wh U'`Uhih\bXUbWY'Uhih\]g''YWh fY']b'h\Y h\YUhfY']g'cdh]cbU''''<ck Yj Yfž'mci 'a i ghYb[U[Y'k]h\'h\Y''YWh fY'UbX'hi hcf]U''dfYdUfUh]cbg'Zcf'k YY\_'cbY' a` \f\W\ZYj Yb']Zmci 'Xc'bch'Uhih\bX'h\Y''YWh fY']b'h\Y'h\YUhfY"

 $\label{thm:linear} H\Y''fYghicZh\Y''YWhifYgiZhca~k~YY\_g'\&'hc'- ž'k~]```VY'cb`]bY'''H\YfY'k~]```VY'bc~YWhifYg']b'h\Y'`YWhifY~h\YUhfY~UZhYf'k~YY\_'cbY''$ 

 $\pm \text{ ``YWhi fYg'mci'k ]```UWhji Y'mYb[U[Y'k]h\'WcfY'Wci fgY'WcbhYbh'h\Uh'k]```\Y`d'mci'hc'UhtU]b'Wci fgY'`YUfb]b[ci hWca Yg'%] +"$ 

#### Hi hcf]U`q

I bYI W gYX'UVgYbW'Zīca 'a cfY'h\Ub'tk c'hi hcf]U`g'k]``fYgi `hi]b'h\Y'Uk UfX'cZU'ZJ]``[fUXY"'?YYd']b'hci W k]h\'mci f'hi hcf'UbX#cf'h\Y'7 ci fgY'7 cbj Ybcf'UVci hmci f'UhhYbXUbW']Zh\YfY'UfY'dfcV`Ya g"'D`YUgY'bchYh\Uh'"=\UX'hc'k cf\_\_"']g'bchfY[UfXYX'Ug'Ub'YI W gY'Zcf'a]gg]b['hi hcf]U`g"

 $\pm$  'hi hcf]U'g'mci 'k ]``'UWhj Y'mYb[U[Y'k]h\'WcfY'Wci fgY'WcbhYbh'h\Uh'k]``'\Y`d'mci 'hc'UhtU]b'Wci fgY'`YUfb]b[ci h\\cappa a Yg'\% +"

#### Ci YfU``

 $D`YUgY`WcbHJWh5bXfYk`ffhY'7cbjYbcfŁ`]Zmci`UfY`\Uj]b[`X]ZJW`hJYg`]b`h\]g`cf`Ubmich\Yf`fYgdYWhk\Yb`]h Wca Yg`hc`mci f`Yb[U[Ya Ybhk]h\YWci fgY"$ 

## 7cifgYGWXYXi`Y

#### J ]Yk 'WUgg'ha YHJV'Y

#### H]a YHJV`Y

8UhY	HmdY	7 cbh/bh
K YY_'%'%': YVfi Ufm!'&( : YVfi Ufm		K \UhXc'k Y'W ffYbhmUggi a Y3'5Vci ha YX]U'UbX Wa a i b]WUhcbg3'5Vci hhYW.bc'c[ m3'5Vci h dYcd'Y3'5Vci hh\Y'k cfX3'5Vci h\ck 'U'``h\YgY Wa Y'hc[Yh\Yf3' <ck 'i="" ``ufy'h\ygy'uggi="" a="" dh]cbg<br="" gyz="">]b'U'W.Ub[]b[ 'k cfX3</ck>
K YY_'&.'&) ': YVfi Ufm!'' A UfVX		H\Y`]bWfYUg]b[`mj_Ughdck_Yfg`cZa_YX]U`UbX Wca_a_i_b]WUf]cbg`]b`YI_dUbXYX`Z]Y`Xg—hYbg]cbg VYfk_YYb`"h\Y`k_cf`X`Ug`a_YX]i_a_"`UbX`h\Y`"h\]fX a_YX]U`fYj_c`i_f]cb"

## FYgci fWyg

## DfYgWf]VYX F Ygci fWYg

 $GYY'7 ci fgY'A ccX'Y'G]hY'Zcf'fYgci fWYg"'H\YfY']g'bc'7 ci fgY'F YUXYf'Zcf'h\]g'Wci fgY''5```fYUX]b[g'UbX ch\Yf'a UhYf]U'gz'cf'`]b\_g'hc'h\YgYz'WUb'VY'Zci bX'cb'h\Y'Wci fgY'A ccX'Y'g]hY''$ 

### FYWca a YbXYX FYgci fW/g

GYY Wci fgY A ccX Y g]hY"

## 7ci fgY 9j U`i Uh]cb UbX 8Yj Y`cda Ybh

7 ci fgY 'ZYYXVUW\_`]g`[ Uh\YfYX`j ]U`X]gW gg]cb`UbX`cVgYfj Uh]cb`Xi f]b[ 'WUggYgžj ]U`Ugg][ ba Ybhk cf\_ž'UbX j ]U`h\Y`A m9I dYf]YbWY`gi fj Ymg"'H\]g`mYUf`h\Y`Wci fgY`\Ug`VYYb`g]a d`]Z]YX`UbX`h\Y`fYUX]b[ g`W hVUW\_`]b fYgdcbgY`hc`ZYYXVUW\_"

Gi Va ]gg]cb cZ5ggYgga YbhHUg\_g

Gi Va ]gg]cb cZ5ggYgga YbhHUg\_g

Hi fb]h]b Gi Va ]gg]cb

\_Zmci YbWci bhYf'U'dfcV`Ya 'k \Yb'UhYa dhJb[ 'hc'gi Va ]hmci f'Ugg][ ba Ybhh\fci [ \ 'Hi fb]hJbž'd`YUgY
hY`Yd\cbY'9| hYfbU``Gi ddcfhcb'-', ) '' ' ' %cf'Ya U]`'h\Ya 'cb'Y! hYfbU`hY`hgi ddcfh4 i bgk "YXi "Ui '"'Gi ddcfh
\ci fg'UfY', .\$\$Ua '-'%\$.\$\$da 'cb'k YY\_XUmg'UbX'-.\$\$Ua '-') .\$\$da 'cb'k YY\_YbXg'fl \*) 'XUmg'U'mYUft'' \( \frac{1}{2}\)mci
UfY'i bUV`Y'hc'gi Va ]hmci f'Ugg][ ba YbhXi Y'hc'U'ZUi 'hk ]h\'Hi fb]hJb'mci 'a Um'Udd'mZcf'Ub'Y! hYbg]cbžVi hmci
a i ghfYhU]b'mci f'hJW\_Yhbi a VYf'Zfca '9! hYfbU`'Gi ddcfhfU'cb[ 'k ]h\'Ubmch\Yf'fY'Yj UbhXcW a Ybhgt'hc
]bWi XY'Ug'Yj ]XYbW'hc'gi ddcfhmci f'Y! hYbg]cb'Udd']\( \frac{1}{2}\)mci 'Ya U]`'9! hYfbU`'Gi ddcfhmci 'k ]``

### 5 WUXYa | W < cbYghmUbX D`U[ | Uf]ga

D`U[]Uf]ga i]gii g]b[ih\Yik cfXgicfi]XYUgicZch\YfgiUbXidfYgYbh]b[ih\Ya iUgimci fick b"i±iWUbihU\_Yia Ubm Zcfa qžZfca iXYi]VYfUhYiW\YUh|b[ihciUVW]XYbhU`mWcdm]b[iZfca iUigci fWik]h\ci hiUW\_bck iYX[Ya Ybh'i

IBGK '[fci dg'd'U[]Uf]ga ']bhc'h\Y'Zc``ck]b[ 'WUhY[cf]Yg.

7 cdm]b[.'i g]b['h\Y'gUa Y'cf'j Yfmg]a]`Uf'k cfXg'hc'h\Y'cf][]bU`hYl h'cf']XYU'k]h\ci h'UW\_bck `YX[]b['h\Y gci fW'cf'i g]b['ei chUh]cb'a Uf\_g"'H\]g'U`gc'Udd`]Yg'hc']a U[Ygž'Ufh'UbX'XYg][b'dfcYWhgž'Ug'k Y``'Ug dfYgYbhUh]cbg'k\YfY'gca YcbY'dfYgYbhg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci h'WfYX]h'

⇒ Uddfcdf]UhY 'dUfUd\fUg]b[...7\Ub[]b[ 'U'ZYk 'k cfXg'UbX'd\fUgYg'k \]'Y 'a cghmfYhJ]b]b[ 'h\Y'cf][]bU` ghfi Wh fY 'UbX#cf'dfc[fYgg]cb'cZ]XYUg'cZh\Y'cf][]bU'žUbX']bZcfa Uh]cb'k ]h\ci hUW\_bck `YX[Ya Ybh'H\]g'U`gc Udd`]Yg']b'dfYgYbhUh]cbg'k \YfY'gca YcbY'dUfUd\fUgYg'Ubch\Yf'g']XYUg'cf'k cfXg'k ]h\ci hWfYX]hUbX'hc d]YW]b[ 'hc[Yh\Yf'ei chYg'UbX'dUfUd\fUgYg']bhc'U'bYk 'k \c`Yžk]h\ci hUddfcdf]UhY'fYZYfYbW]b[ "

7 c``i g]cb. 'k cf\_]b[ 'k ]th 'cth Yfg'Vi hidUgg]b[ 'cZZth Y 'k cf\_'Ug'U'dYfgcb'g']bX]j ]Xi U``k cf\_"'7 c``i g]cb'U`gc ]bWi XYg'dfcj ]X]b[ 'mci f'k cf\_'thc'Ubcth Yf'ghi XYbh'VYZcfY'th Y'Xi Y'XUHYZcf'Zcf'th Y'di fdcgY'cZth Ya

- i bXYfgfUbX mci f'f][\hg'UbX'fYgdcbg]V]`]hJYg'Ug'U'gh XYbhUhl BGK
- VY`Uk UfY`cZd`U[]Uf]ga ž`Wcdmf][\hz'l BGK `Gh XYbh7cXY`cZ7cbXi WhUbX`5VWWdhUV`Y`l gY`cZ I BGK `=\forall H`F Ygci f\W\g`Dc`]\m\m
- VY 'Uk UfY 'cZh\Y 'ghUbXUfXg 'cZVY\Uj ]ci f'Yl dYWYX 'cZYj YfmcbY ']b 'h\Y 'l BGK 'Wca a i b]hm
- `cWUHY 'gYfj ]WYg'UbX ']bZcfa UH]cb'UVci hI BGK 'UbX'I BGK '@]VfUfm

Gca Y'cZh\YgY'UfYUg'k]``VY'ZJa]`]Uf'hc'mci ž'ch\Yfg'k]``VY'bYk"'; U]b]b['U'gc`]X'i bXYfgHJbX]b['cZU``h\YfY`UhYX'UgdYWng'cZ9@+69'k]``\Y`d'mci 'a U\_Y'h\Y'a cghicZmci f'ghi X]Yg'UhI BGK "'\\hd.##qi VYWfi |XYg"`]VfUfm'i bqk "YXi "Ui #\`]qY#JVci hY`]qY

### 5 WUXYa ]W = b Zcfa Uh cb.

: cf`YggYbhJU``gh XYbh`]bZcfa UhJcb`fY`UhJb[ 'hc.

- fYei Yghg Zcf Yl hYbg]cb/
- `UhY 'gi Va ]gg]cbg '[ i ]XY`]bYg/
- fYj ]Yk 'cZa Uf\_g/
- I BGK '<YU`h\ 'UbX'GUZYhm'dc`]W[Yg/
- YI Ua ]bUh]cb dfcW/Xi fYg/
- gdYWJU``Wcbg]XYfUhjcb`]b`h\Y`Yj YbhcZ]``bYgg`cf`a ]gUXj Ybhi fY/
- gh XYbhYei ]hmUbX X]gUV]`]hm/
- UbX'ch\Yf'YggYbhJU`'UWUXYa [W]bZcfa Uh]cbz'gYY

\hdg.#k k k "Uffig"i bgk "YXi "Ui #W ffYbhgh XYbhg#UWUXYa ]W]bZcfa Ufjcb#dfchcWc`g![i]XY`]bYg#

a U[Y'7fYX]h