

\$ on drinks.

3. Line up 40 lots of the number '1' that is

111...111

$\underbrace{\quad}_{40 \text{ lots}}$

Then we can think of this problem as partitioning the list of number 1's into 4 blocks (allowing some block(s) to be empty); The sum of the numbers in each block will be the value of either  $x_1$ ,  $x_2$ ,  $x_3$  or  $x_4$ , and the sum of all blocks will always be 40 because we have a list of 40 lots of '1', i.e  $x_1 + x_2 + x_3 + x_4 = 40$ .

To find the number of ways we can partition the 40 objects into 4 blocks, one can think of it as the number of ways to arrange 43 objects (with 3 walls that separates the 40

objects into blocks). The total number of ways we can do this is  $43!$ , but we don't care about how the objects are arranged because they are all 1's, nor do we care about how the walls are arranged. Hence, the number of solutions is  $43! - 40! - 3! = 12341$ .

- This is a straightforward application of integration by parts [https://en.wikipedia.org/wiki/Integration\\_by\\_parts](https://en.wikipedia.org/wiki/Integration_by_parts). The solution is  $5! = 120$ .
- Let  $r_1; r_2$  be the remainder when the integers  $x_1; x_2$  is divided by some integer  $d$  respectively. One can show that the remainder of  $x_1 + x_2$  divided by  $d$  is  $r_1 + r_2$ , and the remainder of  $x_1 - x_2$  is  $r_1 - r_2$  (write  $x_1; x_2$  in quotient and remainder form, then compute  $x_1 + x_2 / x_1 - x_2$ ).

The remainder of  $1^n$  divided by 7 is always 1.

The remainder of  $2^1$  divided by 7 is 2. Hence, the remainder of  $2^2 = 2 \cdot 2$  divided by 7 is  $2 \cdot 2 = 4$ . Similarly, the remainder of  $2^3 = 2^2 \cdot 2$  divided by 7 is  $4 \cdot 2 = 8$ , which is the same as 1. Finally, the remainder of  $2^4 = 2^3 \cdot 2$  divided by 7 is  $1 \cdot 2 = 2$ . Since the remainder of  $2^4$  divided by 7 is the same as  $2^1$ , if we keep increasing the power  $n$  in  $2^n$  we would expect the remainders to be 2; 4; 1 repeating, because we are multiplying the remainder by the same factor of 2 each time.

Analogously, the remainders of  $4^n$  divided by 7 are 4; 2; 1 repeating.

Therefore, the remainder of  $1^n + 2^n + 4^n$  divided by 7 is 1 + 2 + 4; 1 + 4 + 2; 1 + 1 + 1 repeating, which is the same as 0; 0; 3 repeating. Thus, we can argue that 2/3 of 999 the numbers between  $0 < n < 1000$  makes  $1^n + 2^n + 4^n$  divisible by 7.

- Draw the line  $BO$ , where  $O$  is the origin of the circle; see below. From the figure, we can easily find  $x$  using Pythagoras on the 2 right angled triangles. The length of  $BC$  is twice that of  $x - 2$ .



